Potential Solution of a Homogeneous Strip-Line of Finite Width*

W. H. HAYT, JR.†

Summary—**The exact potential solution for a zero-thickness strip centered between two ground planes of finite width is outlined. For unit separation of the ground planes, the solution is applied to obtain curves of capacitance per unit length for several representative ground plane widths as a function of strip width. The results are valid for all strip widths, including the case in which the strip is wider than the ground planes. The validity of assuming infinite ground plane width is investigated and it is found that such an assumption leads to little error providing the ratio of ground plane width to separation is at least 2.5, and also providing the difference between ground plane and strip width is at least one-half of the ground plane separation.**

Introduction

Included in the several varieties of microstrip transmission lines is the strip between two parallel ground planes, commonly called strip-line.[1–3] It is usually sufficient to consider the width of the ground planes to be infinite, for in most practical cases they are

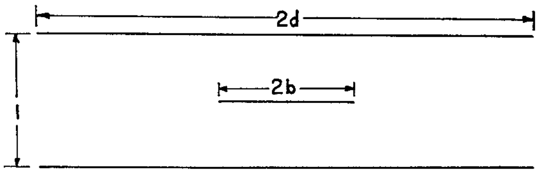

Fig. 1—Strip-line composed of zero-thickness strip centered between two parallel zero-thickness ground planes of finite width.

carefully chosen to be sufficiently wide to inhibit radiation leakage and mutual coupling between adjacent lines. It is the purpose of this article to investigate the validity of this infinite width assumption by obtaining the exact solution for the strip-line shown in Fig. 1. The strip and both ground planes are of zero thickness.

The Conformal Transformation

Fig. 2 illustrates the geometry of the line cross section in the z-plane, and Figs. 3, 4, and 5 (opposite page), respectively, show the auxiliary t- and r-planes, and the $W=\phi+j\psi$ complex potential plane.

The mapping from the z-plane to the t-plane satisfies the differential equation,

$$dz/dt = C_1(t-D)/\sqrt{(t-B)(t-C)(t-E)},$$

where, $B=0$, $C=1$, $D=1/k_1^2$, $E=1/k^2$, $k<k_1<1$. Letting $t=\sin^2\beta$, the answer appears in terms of elliptic integrals of the first and second kinds, as

$$z = \frac{2C_1}{k}\left[(1-k^2/k_1^2)F(k,\beta) - E(k,\beta)\right] + C_2.$$

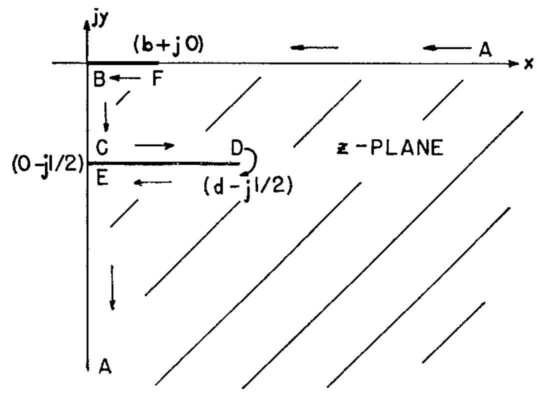

Fig. 2—The lower right quadrant of the strip-line in the complex z-plane. Arrows indicate direction of traverse around boundary of polygon with shaded interior.

Applying the boundary condition at $z=-j1/2$, $t=1$, $\beta=\pm\pi/2$ gives the value of C_1,

$$4C_1 = -jk/\left[(1-k^2/k_1^2)K - E\right]$$

and letting $z=-j1/2$, $t=1/k^2$, $\beta=\sin^{-1}1/k$, we find

$$k^2/k_1^2 = E'/K'$$

allowing us to write

$$z = j\frac{K'}{\pi}\left[(1-E'/K')F(k,\beta) - E(k,\beta)\right].$$

At the edge of the guard plane, $z=d-j1/2$, $t=1/k_1^2$, $\beta=\sin^{-1}1/k_1$, which, upon substitution and simplification of the resultant elliptic integrals, reduces to

$$d = \frac{K'}{\pi}\left[E(k',\gamma) - \frac{E'}{K'}F(k/k',\gamma)\right],$$

* This paper is based on a portion of a Ph.D. thesis, written at the University of Illinois, entitled "A Study of Radiation and Mutual Impedance Problems in Strip Transmission Lines."

† School of Elec. Engrg., Purdue Univ., Lafayette, Ind.

[1] R. M. Barrett, "Etched sheets serve as microwave components," *Electronics*, vol. 25, pp. 114–118; June, 1952.

[2] D. D. Grieg and H. F. Engelmann, "Microstrip—a new transmission technique for the kilomegacycle range," Proc. IRE, vol. 40, pp. 1644–1650; December, 1952.

[3] E. G. Fubini, W. Fromm, and H. Keen, "New techniques for high-Q microwave components," 1954, IRE Convention Record, *Part 8*, "Communications and Microwave," pp. 91–97.

where,

$$\sin^2 \gamma = (1 - E'/K')/k'^2.$$

At the edge of the strip, a similar method leads to

$$b = \frac{K'}{\pi}\left[\frac{E'}{K'}F(k', \delta) - E(k', \delta) + \frac{1}{k_d}\sqrt{k_d^2 + k^2}/\sqrt{k_d^2 + 1}\right]$$

where,

$$\tan \delta = 1/k_d.$$

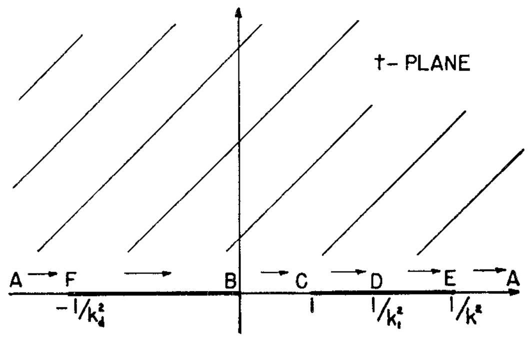

Fig. 3—The auxiliary t-plane. The real axis comprises the boundary of the polygon described in the z-plane.

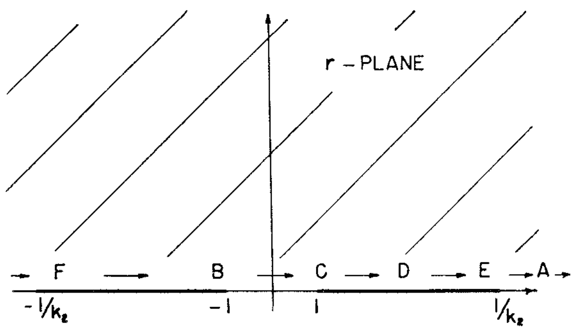

Fig. 4—The auxiliary r-plane. The real axis is again the boundary of the polygon but it has been distorted by a bilinear transformation.

The transformation from the t-plane to the r-plane is by a bilinear transformation, and the W-plane and r-plane are again related by an elliptic integral. The various transformations are collected below, along with the relationships between the dimensional constants and the several functions.

$$z = j\frac{K'}{\pi}\left[(1 - E'/K')F(k, \beta) - E(k, \beta)\right] \quad (1)$$

$$t = \sin^2 \beta \quad (2)$$

$$k' = \sqrt{1 - k^2} \quad (3)$$

$$K' = K(k') \qquad E' = E(k') \quad (4)$$

$$d = \frac{1}{\pi}\left[K'E(k', \gamma) - E'F(k', \gamma)\right] \quad (5)$$

$$\sin^2 \gamma = (1 - E'/K')/k'^2 \quad (6)$$

$$b = \frac{K'}{\pi}\left[\frac{E'}{K'}F(k', \delta) - E(k', \delta) + \frac{\sqrt{k_d^2 + k^2}}{k_d\sqrt{k_d^2 + 1}}\right] \quad (7)$$

$$\tan \delta = 1/k_d \quad (8)$$

$$t = \frac{r + 1}{r(2 + a_0) - a_0}, \qquad r = \frac{ta_0 + 1}{t(2 + a_0) - 1} \quad (9)$$

$$a_0 = \frac{k^2(1 + k_2) - 2}{1 - k_2} \quad (10)$$

$$k_2 = (k^2 + k_d^2)/\left[2 + k_d^2 - k^2 + 2\sqrt{(1 - k^2)(1 + k_d^2)}\right] \quad (11)$$

$$W = \frac{\phi_0}{2}\left[\frac{F(k_2, \alpha)}{K(k_2)} + 1\right], \qquad \sin \alpha = r \quad (12)$$

$$\phi_0 = K(k_2)/K(k_2') \quad (13)$$

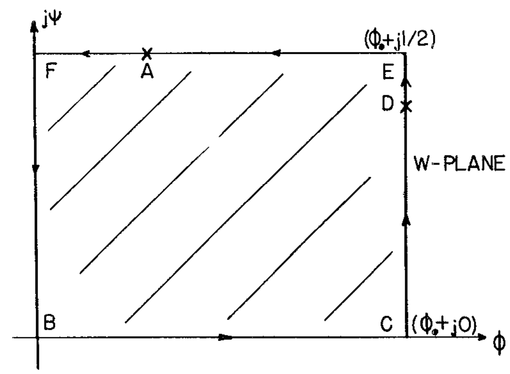

Fig. 5—The complex potential or W-plane. The potential and stream functions are respectively ϕ and ψ.

The equations relating the dimensions b and d to k and k_d are transcendental and cannot be solved explicitly.

Capacitance

The potential difference between strip and ground planes is seen to be ϕ_0 from Fig. 5 and the total charge on the strip is $2\epsilon_0$. We then have

$$C = 2\epsilon_0/\phi_0 = 2\epsilon_0 K(k_2')/K(k_2). \quad (14)$$

The capacitance per unit length for a strip-line of given dimensions cannot be found directly because of the implicit nature of the equations above. Useful information is, however, presented by a family of curves which may be computed with little labor. The computational process begins with a choice of k, from which k', K, K', γ, $E(k', \gamma)$, and $F(k', \gamma)$ are derived by (3), (4), and (6). The guard plane half-width, d, is then determined by (5). After choosing k_d, then δ, $F(k', \delta)$, and $E(k', \delta)$ are obtained by (8), which now defines b, the strip half-width, from (7). Using (11), k_2 is readily found, from which $K(k_2)$, $K(k_2')$, and in turn C are determined.

Fig. 6 shows C, the capacitance of the strip-line per unit length, as a function of b, the half-width of the strip, for several values of d, the half-width of the ground planes. For values of b less than 1.00, the curves for d having values of 1.250 and infinity agree to about one-quarter of one per cent. The asymptotic values arise from exact expressions for the capacitance of parallel plate condensers.

It therefore follows that for $d>1.25$ and $d>b+0.25$, the exact capacitance and approximate value obtained by assuming infinite width ground planes agree within one-quarter of one per cent. A similar statement may be made for the characteristic impedance, since $R_0=\sqrt{\mu_0\epsilon_0}/C$. The approximation is also valid for smaller values of d providing a more severe restriction is placed on the magnitude of b.

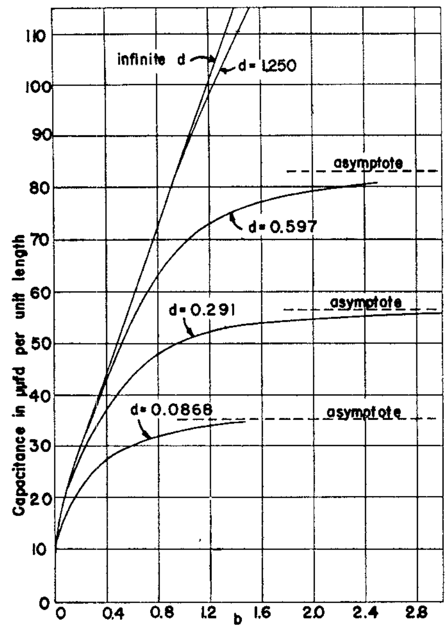

Fig. 6—Capacitance per unit length for a strip-line having a strip of width $2b$ centered between two ground planes each of width $2d$. The ground planes are separated by unit distance.

Resonant Frequencies of Higher-Order Modes in Radial Resonators

D. C. STINSON†

Summary—**A summary of the relevant work on radial line discontinuities and radial line resonators is presented. A step-type discontinuity is analyzed using an integral equation formulation and the results are applied to the calculation of the resonant frequencies of a radial resonator. This method is verified by experiment and compared with the foreshortened-line approximation and with the methods of Marcuvitz and Goddard, whose work is satisfactory for the lowest-order TM mode. However, the present method is the only one which is equally applicable to the calculation of the resonant frequencies of TM modes possessing higher-order radial variations.**

Introduction

RADIAL LINE discontinuities have been considered quite completely by Whinnery[1] and by Bracewell.[2] The former presents his data in the form of curves and takes into account such factors as the proximity of a shorting cylinder near the discontinuity and the effect of higher-order, nonpropagating modes. The latter treats only the simple step discontinuity, with the step facing either the inner region or the outer region, and takes into account radial variations by a cylindrical spread factor which is given by families of curves. These two papers form a very complete picture of radial line discontinuities.

Radial resonators have been considered by several approximate methods,[3–5] but these ignore the discontinuity capacitance. Ordinarily, the discontinuity capacitance is of the same order of magnitude as the capacitance of the capacity loading of the gap, region A, in Fig. 1, if one considers the resonator as a foreshortened radial or coaxial line resonator. A better method considers the modes in both regions and matches them across the aperture, $r=a$. This method has been used

† Elect. Res. Lab., Univ. of Calif., Berkeley, Calif.

[1] J. R. Whinnery, "Radial line discontinuities," Elec. Lab., General Electric Co., D. F. #46293; June 22, 1944.
J. R. Whinnery and D. C. Stinson, "Radial line discontinuities," Proc. IRE, vol. 43, pp. 46–51; January.
[2] R. N. Bracewell, "Step discontinuities in disk transmission lines," Proc. IRE, vol. 42, pp. 1543–1548; October, 1954.
[3] F. E. Terman, "Radio Engineers' Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., p. 268; 1943.
[4] S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio," John Wiley and Sons, Inc., New York, N. Y., pp. 404–412; 1944.
[5] J. C. Slater, "Microwave Electronics," D. Van Nostrand Co., Inc., New York, N. Y., p. 234; 1950.